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Tangential formal deformations of the Poisson bracket and tangential star products on a regular Poisson manifold

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We provide the existence of tangential formal deformations of the Poisson bracket on a regular Poisson manifold. We study relations between these deformations and tangential star products. We deduce an existence theorem for these star products.

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Introduction

Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer introduced in ref. [1] the notion of a star product (deformation of associative and Lie algebra structures on the space of C^∞ functions on a manifold M) in order to give a precise mathematical definition of quantization for a classical mechanical system. The questions of the existence and equivalence of such star products were essentially studied for symplectic manifolds. Using a cohomological computation of Gutt [4,5], De Wilde and Lecomte in ref. [3] proved the existence and studied the equivalence for any symplectic manifold. However, for many physical problems, the natural structure is a Poisson manifold (for instance, for time dependent dynamical systems); thus the problem of a star product on a Poisson manifold was introduced. For instance, in ref. [2], Lichnerowicz defines tangential star products on regular Poisson manifolds; that notion is natural because of the following facts:

(1) The star product is an algebraical deformation of a structure of an associative and Lie algebra on the space of C^∞ functions on M. Thus the natural object is the Poisson bracket, characteristic of the Poisson manifold structure on M and not a symplectic two-form. Moreover, the deformation theory of the Lie algebra structure on $C^\infty(M)$ for the Poisson bracket is based on cohomology groups $H^n(C^\infty(M), \partial)$. In ref. [2], Lichnerowicz proved the existence of the tangential

star product on a regular Poisson manifold if $H^3(C^\infty(M), \partial)$ vanishes, which is the first step of such a theory of deformations.

(2) From the geometrical point of view, if M is symplectic, the existence theorem uses heavily the cohomology classes of S^3_Γ and T^2_Γ [1] (see ref. [3] for the definition). These classes are canonical geometrical objects for M. But their definition requires only the structure of a regular Poisson manifold on M [2] and the natural set-up of the theory is thus the category of such manifolds.

Finally, let us mention the thesis of Guédira [6], where the existence of tangential star products is proved for a tangentially exact regular Poisson manifold; however, her cohomological computation is not completely correct, thus we give here the cohomology groups we need (proposition 6.5).

In this article, we generalize the method used in ref. [3]. That is, first we build Θ^τ_Λ, which is a two-cocycle for the cohomology of $\mathbb{Z}$-graded Lie algebras [9], and then we deduce formal deformations L_ν of the Poisson bracket, solution of the equation:

$$(\nu\,\partial/\partial\nu+1)L_\nu+\tfrac{1}{2}\Theta^\tau_\Lambda(L_\nu, L_\nu)=0\,.$$

The cocycle Θ^τ_Λ is defined in the symplectic case with the use of differential forms on the manifold. This is no longer possible in our context, thus we define here Θ^τ_Λ from contravariant tensors on M; these are the only natural objects in this theory. In fact because these canonical objects are all tangential, our deformations are naturally tangential.

In the first three sections we give definitions of graded Lie algebras and deformations, and local notation; in sections 4 and 5 we define tangential star products and (without proof) quote easy generalizations of some results of ref. [3] for regular Poisson manifolds. The computation of tangential Chevalley cohomology spaces is performed in detail in section 6; indeed the formulation of our result requires applications from spaces of one-forms instead of maps from spaces of vector fields. We define Θ^τ_Λ and study its properties in section 7. Formally our propositions are exactly the same as those of ref. [3] but the proofs are very different and we give them completely here. We prove the existence results in sections 8 and 9, which are completely similar to the corresponding results of ref. [3].

1. Graded Lie algebras associated with a vector space

Let V be a vector space; we denote by $\mathscr{M}^a(V)$ the space of all $(a+1)$-linear maps from V into V and by $\mathscr{A}^a(V)$ the space $\alpha(\mathscr{M}^a(V))$, α is the antisymmetrization projector,

$$\alpha(A)(x_0, ..., x_a)=\frac{1}{(a+1)!}\sum_{\sigma\in P_{a+1}} \operatorname{sign}(\sigma)\, A(x_{\sigma(0)}, ..., x_{\sigma(a)})\,,$$

where P_{a+1} is the group of permutations of $\{0, ..., a\}$ and $\operatorname{sign}(\sigma)$ is the signature of σ. We put

$$\mathcal{M}(V)=\bigoplus_{a\geqslant -1}\mathcal{M}^a(V)\,,\qquad \mathcal{A}(V)=\bigoplus_{a\geqslant -1}\mathcal{A}^a(V)\,.$$

We define

$$\Delta:\mathcal{M}(V)\times\mathcal{M}(V)\to\mathcal{M}(V)\,,$$

$$A\Delta B=i(B)A+(-1)^{ab+1}i(A)B\quad\forall A\in\mathcal{M}^a(V),\ \forall B\in\mathcal{M}^b(V)\,,$$

where

$$i(B)A=0\quad\text{if}\ A\in\mathcal{M}^{-1}(V)\,,$$

$$i(B)A(x_0,...,x_{a+b})=\sum_{k=0}^{a}(-1)^{kb}A(x_0,...,B(x_k,...,x_{k+b}),...,x_{a+b})$$

$$\text{if}\ A\in\mathcal{M}^a(V)\ (a>-1),\ B\in\mathcal{M}^b(V)\,.$$

We also define

$$[\ ,\ ]:\mathcal{A}(V)\times\mathcal{A}(V)\to\mathcal{A}(V)\,,$$

$$[A,B]=\frac{(a+b+1)!}{(a+1)!(b+1)!}\alpha(A\Delta B)\,,\quad\forall A\in\mathcal{A}^a(V),\ \forall B\in\mathcal{A}^b(V)\,.$$

It is easy to see that $(\mathcal{M}(V),\Delta)$ and $(\mathcal{A}(V),[\ ,\ ])$ are $\mathbb{Z}$-graded Lie algebras. Moreover, if $A\in\mathcal{M}^1(V)$, then (V,A) is an associative algebra if and only if $A\Delta A=0$ and a Lie algebra if and only if $A\in\mathcal{A}^1(V)$ and $[A,A]=0$.

Proposition 1.1 [3]. *Let $(E,\circ)$ be a $\mathbb{Z}$-graded Lie algebra, $E=\oplus_{n\in\mathbb{Z}}E^n$. If $A\in E^1$ is such that $A\circ A=0$ then $\partial_A:E\to E$, $B\in E^b\to(-1)^bA\circ B$, is a homogeneous map with degree 1 satisfying*

$$\partial_A\circ\partial_A=0\,,\ \partial_A(B\circ C)=(-1)^c(\partial_AB)\circ C+B\circ(\partial_AC)\,,\quad\forall B\in E\,,\ \forall C\in E^C\,.$$

Thus, $\circ$ induces on the cohomology space $H(E,\partial_A)=\ker\partial_A/\operatorname{Im}\partial_A$ a $\mathbb{Z}$-graded Lie algebra structure.

2. Formal deformation

(a) Let V be a vector space. We denote by V_ν the space of all formal series

$$x_\nu=\sum_{i\geqslant 0}\nu^ix_i\,,\quad x_i\in V\,.$$

Definition. An element A_ν of $\mathcal{M}^a(V_\nu)$ is *formal* if it has the form

$$A_\nu : (x_\nu^{(0)}, \ldots, x_\nu^{(a)}) \to \sum_{i \geqslant 0} \nu^i \left(\sum_{r+s_0+\cdots+s_a=i} A_r(x_{s0}^{(0)}, \ldots, x_{sa}^{(a)}) \right),$$

where $A_r \in \mathcal{M}^a(V)$.

A_r is called the rth component of A_ν; this can be written $A_\nu = \sum_{i \geqslant 0} \nu^i A_i$, so A_ν is identified with an element of $(\mathcal{M}(V))_\nu$. It is also easy to see that $\mathcal{M}(V)_\nu$ [respectively $\mathcal{A}(V)_\nu$] is a graded Lie sub-algebra of $(\mathcal{M}(V_\nu), \Delta)$ [respectively $\mathcal{A}(V_\nu), [\ ,\ ])$].

(b) Let (V, A) be an associative or a Lie algebra.

Definition

(1) A *formal deformation* A_ν of A is an associative or Lie algebra structure on V_ν such that A_ν is formal and $A_0 = A$.

(2) Let us write $\circ$ for Δ or $[\ ,\ ]$. A *formal deformation of order k* of (V, A) is a formal A_ν such that $A_0 = A$ and $\sum_{i+j=l} A_i \circ A_j = 0$ $\forall l \leqslant k$.

Proposition 2.1 [3]. *A bilinear formal map $A_\nu = \sum_{i \geqslant 0} \nu^i A_i$ is a formal deformation of order k of A_0 if and only if*

$$2\partial_{A_0} A_i = J_i \ \forall i \leqslant k, \quad J_i = \sum_{r+s=i,\, r,s>0} A_r \circ A_s;$$

in this case we have $\partial_{A_0} J_{k+1} = 0$.

(c) **Definition.** Two formal deformations A_ν and A'_ν of (V, A) are said to be *formally equivalent* if and only if there exists

$$T_\nu = \sum_{i \geqslant 0} \nu^i T_i \in \mathcal{M}^0(V)_\nu$$

such that $T_0 = 1$ and $A'_\nu = T^*_\nu(A_\nu)$, where

$$T^*_\nu(A_\nu)(x_\nu, y_\nu) = T_\nu(A_\nu(T_\nu^{-1}(x_\nu), T_\nu^{-1}(y_\nu))) .$$

They are said to be *formally equivalent up to order k* if and only if the components of $T^*_\nu(A_\nu)$ and A'_ν are equal up to order k.

3. Local maps and symbols

In the sequel of this article, M denotes a smooth connected and second countable manifold. Let E and F be two vector bundles on M; we denote by $\Gamma(E)$ and $\Gamma(F)$ the space of smooth sections of E and F, respectively.

Definition [7]. A multilinear map $C:\Gamma(E)^{c+1}\to\Gamma(F)$ is *local* if and only if

$$\operatorname{supp} C(s_0, ..., s_c) \subset \bigcap_{i=0}^{c} \operatorname{supp} s_i \quad \forall s_0, ..., s_c \in \Gamma(E) ,$$

where supp s is the support of s.

It is well known [7] that locally C is a multilinear differential operator. So, for any relatively compact domain U of a chart on M, and for any local coordinates $(x^1, ..., x^n)$ on U, if we give trivializations of E and F over U, we can write

$$C(s_0, ..., s_c)_{/x} = \sum A_{\alpha_0,...,\alpha_c,x}(D_x^{\alpha_0}\bar{s}_0, ..., D_x^{\alpha_c}\bar{s}_c) , \quad \forall s_0 ..., s_c \in \Gamma(E) ,$$

where $\bar{s}_i$ is the local form of s_i and where $A_{\alpha_0,...,\alpha_c,x}$ is a $(c+1)$-linear map from E_0, the typical fiber of E, into F_0, the typical fiber of F. Moreover, the sum is finite.

Definition [7]. A local map is said to be *k-differentiable* if and only if its restriction to any domain of a natural chart is given by a multidifferential operator of maximal order k in each argument. It is said to be *differentiable* if and only if it is k-differentiable for some integer k.

It is also well known [7] that, if C is differentiable and has total order $s=\sup\sum_{i=0}^{c}|\alpha_i|$ such that $A_{\alpha_0,...,\alpha_c}\neq 0$ on some U, then

$$\sigma_C(\xi_0, ..., \xi_c) = \sum_{|\alpha_0|+\cdots+|\alpha_c|=s} (\xi_0)^{\alpha_0}\cdots(\xi_c)^{\alpha_c} A_{\alpha_0,...,\alpha_c,x}, \quad \xi_i \in T_x^*M ,$$

is an intrinsically defined map called the total symbol of C, and if $(r_0, ..., r_c)$ is the maximum in the lexicographical order of the $(c+1)$-tuples $(|\alpha_0|, ..., |\alpha_c|)$ such that $|\alpha_0|+\cdots+|\alpha_c|=s$ and $A_{\alpha_0,...,\alpha_c}\neq 0$ on U, then

$$\sigma_C(\xi_0, ..., \xi_c) = \sum_{|\alpha_i|=r_i} (\xi_0)^{\alpha_0}\cdots(\xi_c)^{\alpha_c} A_{\alpha_0,...,\alpha_c,x}$$

is an intrinsically defined map called the lexicographical symbol of C, and $(r_0, ..., r_c)$ is called the lexicographical order of C.

4. Tangential star products and tangential formal deformations of (N, P)

Suppose now that M is Poisson manifold, so it is provided with a structure tensor Λ (cf. ref. [2] for definitions and notations). We denote by N the source of all smooth functions over M. The Poisson bracket on N is defined by

$$[u, v] = P(u, v) = \Lambda(\mathrm{d}u, \mathrm{d}v) \quad \forall u, v \in N .$$

N is an associative algebra for the usual multiplication $m:(u, v)\to uv$ and a Lie algebra for P. We denote by I the centre of this Lie algebra. The elements of I are functions f such that

$$\{u,f\}=0 \quad \forall u\in N\,.$$

It can be written as

$$H_u f=0 \quad \forall u\in N\,,$$

where H_u is the Hamiltonian vector field associated with u (cf. ref. [2]); such an f will be called an invariant.

Definition [3]. A *weak star product* is a formal deformation of m,

$$M_\nu = m+\nu P+\sum_{k\geqslant 2}\nu^k M_k\,,$$

such that

(i) $M_k(u,v)=(-1)^k M_k(v,u)\ \forall u,v\in N$,
(ii) M_k is local,
(iii) M_k is nc (vanishing on constants) if k is odd.

It will be said to be a *star product* if M_k is nc for each $k>1$.

In the sequel of this note we suppose that (M,Λ) is regular and we denote by $2p$ the dimension of leaves and by q their codimension. We provide M [2] with an atlas of natural charts such that the only nonvanishing components of Λ are

$$\Lambda^{i,i+p}=-\Lambda^{i+p,i}=1\,,\quad i\in\{1,\dots,p\}\,.$$

Definition [2]. Let Γ be a connection without torsion on M, and let us consider the corresponding connection one-form $\omega^i_j=\Gamma^i_{j,k}\mathrm{d}x^k$. Γ is said to be adapted to the leaves if and only if $\Gamma^a_{i,k}=0\ \forall a>2p,\ \forall i\leqslant 2p$.

Let ∇ be the covariant derivative associated with Γ, and let C be a local $(r+1)$-linear map from N into N, so for any relatively compact domain U of a chart in M there exists a family of contravariant tensors

$$T_{k_0,\dots,k_r}\,,$$

symmetric in their arguments $k_0+\dots+k_i+j$ $(1\leqslant j\leqslant k_{i+1})$, such that

$$C(u_0,\dots,u_r)=\sum_{k_0,\dots,k_r}\langle T_{k_0,\dots,k_r},\nabla^{k_0}u_0\otimes\dots\otimes\nabla^{k_r}u_r\rangle\,.$$

Definition [2]. C is said to be *tangential* if and only if all the tensors $T_{k_0,\dots,k_r}$ are tangential.

Remark [2]. This definition is independent of the choice of Γ.

Let E and F be two vector bundles on M with typical fibers E_0 and F_0, having two bases $(e_i)_{i\in I}$ and $(f_j)_{j\in J}$, and let C be a local $(r+1)$-linear map from $\Gamma(E)$ into $\Gamma(F)$, so that there exists a family of $(r+1)$-linear maps $(C_j)_{j\in J}$ from $\Gamma(E)$ into N such that

$$C(s_0, \ldots, s_r) = \sum_{j\in J_0} C_j(s_0, \ldots, s_r)f_j \quad (J_0 \text{ finite}), \quad \forall s_0, \ldots, s_r \in \Gamma(E).$$

If for $i_0, \ldots, i_r \in I$ we put

$$C_j^{i_0,\ldots,i_r}(u_0, \ldots, u_r) = C_j(u_0 e_{i_0}, \ldots, u_r e_{i_r}) \quad \forall u_0, \ldots, u_r \in N,$$

then

$$C(s_0, \ldots, s_r) = \sum_{j\in J_0} \sum_{i_0\in I_0,\ldots,i_r\in I_r} C_j^{i_0,\ldots,i_r}(s_0^{i_0}, \ldots, s_r^{i_r})f_j, \quad \text{if } s_k = \sum_{i_k\in I_k} s_k^{i_k} e_{i_k}.$$

Definition. C is said to be *tangential* if and only if $\forall j\in J$ and $\forall i_0, \ldots, i_r \in I$, $C_j^{i_0,\ldots,i_r}$ is tangential.

Definition [2]. A *Poisson connection* is a connection without torsion such that $\nabla\Lambda = 0$.

Remark [2]. Every Poisson connection is adapted to the leaves.

Definition. Let A_0 be a tangential two-linear map from N into N defining on N an associative or Lie algebra structure. A formal deformation

$$A_\nu = \sum_{k\geq 0} \nu^k A_k$$

of A_0 is said to be *tangential* if and only if A_k is tangential for each k.

If $M_\nu = \sum_{k\geq 0} \nu^k M_k$ is a weak star product, then $P_\nu = \sum_{k\geq 0} \nu^k M_{2k+1}$ is a formal deformation of P; we say that P_ν derives from M_ν.

Using the same argument as in the symplectic case [3] we obtain

Lemma 4.1. *Let $A \in M^1_{\mathrm{loc}}(N)$; then*

(i) $m\triangle A$ is nc if and only if $A = A' + am$, where $A' \in M^1_{\mathrm{loc,nc}}(N)$ and $a \in N$;

(ii) $P\triangle A = 0$ if and only if $A = am$, where $a \in I$. □

As in the symplectic case [3], we deduce the following two propositions:

Proposition 4.2. *Every tangential weak star product is formally equivalent to a tangential star product.* □

Proposition 4.3. *A tangential formal deformation of P cannot derive from several tangential star products: the map*

$$M_v = \sum_{k\geq 0} v^k M_k \to \sum_{k\geq 0} v^k M_{2k+1}$$

is injective. □

5. Tangential Hochschild cohomology of (N, m) and the term M_2 of a tangential star product

The Hochschild cohomology operator on (N, m) is defined by $\delta A = (-1)^a m \triangle A$ $\forall A \in \mathscr{M}^a(N)$. It is clear that the graded subspace $\mathscr{M}_{\mathrm{loc,t}}(N)$ of all tangential elements of $\mathscr{M}(N)$ is stable under δ. We denote by $H_{\mathrm{t}}(N, \delta)$ the cohomology group corresponding to this subspace.

Proposition 5.1 [2]. *For $k=2$ or 3 the space $H_{\mathrm{t}}^k(N, \delta)$ is isomorphic to the space of all tangential contravariant skew-symmetric tensors on M.* □

Let Γ be a Poisson connection and denote by ∇ the covariant derivative associated with Γ. We define the two-differential map P_Γ^2 by

$$P_\Gamma^2(u, v)_{/U} = \Lambda^{i,k} \Lambda^{j,l} \nabla_{i,j} u \nabla_{k,l} v , \quad \forall u, v \in N .$$

Proposition 5.2 [2]. *The term M_2 of a tangential star product M_v of order ≥ 2 has the form*

$$M_2 = \tfrac{1}{2} P_\Gamma^2 + \delta T , \; T \in M_{\mathrm{loc,t}}^0(N) .$$

Since $\alpha(P \triangle M_2) = 0$, $P \triangle M_2$ is a coboundary, thus there exists M_3 such that $m + vP + v^2 M_2 + v^3 M_3$ is a tangential star product of order 3. To obtain the term of order 4 it is necessary that $\alpha(P \triangle M_3 + M_2 \triangle M_2) = 0$, thus $[P, M_3] = 0$, and M_3 is a cocycle for the tangential Chevalley cohomology of (N, P). □

6. Tangential Chevalley cohomology of (N, P)

Given two vector spaces E, F we denote by $\Lambda^r(E, F)$ the space of all r-linear alternating maps from E into F and we write

$$\Lambda(E, F) = \bigoplus_{r \geq 0} \Lambda^r(E, F) .$$

We denote by $H(M)$ the space of all smooth vector fields over M and by $H_{\mathrm{t}}(M)$

the subspace of tangential elements of $H(M)$. The space of smooth r-forms is denoted by $\Lambda^r(M)$ and we write

$$\Lambda(M) = \bigoplus_{r \geqslant 0} \Lambda^r(M) .$$

The Chevalley coboundary operator ∂ of the adjoint representation of (N, P) is

$$\partial A = (-1)^a [P, A] \quad \forall A \in \mathscr{A}^a(N) .$$

It stabilizes the spaces $\mathscr{A}_{\mathrm{loc,t,nc}}(N)$ (the space of tangential local maps vanishing on the constants from N into N). We denote by $H_{\mathrm{loc,t,nc}}(N, \partial)$ the corresponding cohomology.

In order to compute the first two groups of this cohomology, we exhibit some particular cocycles, following the same way as in the symplectic case. The two-tensor Λ defines a morphism

$$\rho : \Lambda^1(M) \to H(M) ,$$

which extends naturally to a map from $A(H(M), \Lambda(M))$ into $A(\Lambda^1(M), \Lambda(M))$. We define the map

$$\rho^* : A(\Lambda^1(M), N) \to \mathscr{A}_{\mathrm{nc}}(N) ,$$

$$\rho^* T(u_0, \ldots, u_{r-1}) = T(\mathrm{d}u_0, \ldots, \mathrm{d}u_{r-1}) \quad \forall T \in A^r(\Lambda^1(M), N) ,$$

and we write $\mu^* = \rho^* \circ \rho'$, where ρ' is the restriction of ρ to $A(H(M), N)$. The Lie derivative on the space of smooth forms $\Lambda(M)$ of M is a representation of $(H(M), [\ ,\ ])$. The corresponding differential is denoted by ∂'.

Proposition 6.1 [3].

$$\mu^* \circ \partial'(C) = \partial \circ \mu^*(C) \quad \forall C \in A(H(M), N) . \qquad \square$$

Proposition 6.2. *For any $X \in H_t(M)$ there exists $\omega \in \Lambda^1(M)$ such that $X = \rho(\omega)$.*

Proof. Let $(U, x^1, \ldots, x^n)$ and $(U', y^1, \ldots, y^n)$ be two natural charts of M such that $U \cap U' \neq \emptyset$. If

$$X_{/U} = \sum_{i=1}^{2p} X_i \frac{\partial}{\partial x^i} , \qquad X_{/U'} = \sum_{i=1}^{2p} X'_i \frac{\partial}{\partial y^i} ,$$

then we write

$$\omega_U = \sum_{i=1}^{2p} (-1)^{\chi_p(i)+1} X_{s(i)} \, \mathrm{d}x^i ,$$

$$\omega_{U'} = \sum_{i=1}^{2p} (-1)^{\chi_p(i)+1} X'_{s(i)} \, \mathrm{d}y^i ,$$

where χ_p is the characteristic function of $\{1, ..., p\}$ and $s(i)=i+p$ if $i\leqslant p$, $s(i)=i-p$ if $i>p$. They are defined on U and U' and $\rho(\omega_U)=X_{/U}$ and $\rho(\omega_{U'})=X_{/U'}$. Moreover, if $\mathrm{d}y^i=\sum_{j=1}^{n} f_{i,j}\,\mathrm{d}x^i$ over $U\cap U'$, then from the relations

$$\sum_{i=1}^{2p} X_i\,\mathrm{d}x^i=\sum_{i=1}^{2p} X'_i\,\mathrm{d}y^i\,,$$

$$\sum_{j=1}^{2p}(-1)^{\chi_p(j)} f_{k,j} f_{r,s(j)}=\Lambda(\mathrm{d}y^k,\mathrm{d}y^r)=\begin{cases}(-1)^{\chi_p(k)} & \text{if } r=s(k)\,,\\ 0 & \text{otherwise}\,,\end{cases}$$

on $U\cap U'$, $k, r\in\{1, ..., 2p\}$, we deduce that $\omega_U=\omega_{U'}$ on $U\cap U'$. □

We put $\mathbb{B}=\rho(\Lambda(H(M),\Lambda(M)))$ and we define on $\mathbb{B}$ a cohomology operator ∂'' by

$$\forall T\in\mathbb{B}\quad \text{if } T=\rho(C)\quad \text{then } \partial''T=\rho(\partial' C)\,.$$

This definition is independent of the choice of C. We denote by $\Lambda_{\mathrm{loc,t,nt}}(\Lambda^1(M), \Lambda(M))$ the space of all tangential multilinear maps from $\Lambda^1(M)$ into $\Lambda(M)$ vanishing on the "transversal" forms, the elements of T^*M vanishing on $H_t(M)$.

Proposition 6.3.

(i) $\Lambda_{\mathrm{loc,t,nt}}(\Lambda^1(M),\Lambda(M))\subset\mathbb{B}$,

(ii) $\mathscr{A}_{\mathrm{loc,t,nc}}(N)=\rho^*(\Lambda_{\mathrm{loc,t,nt}}(\Lambda^1(M),N))$.

Proof.

(i) Let $T\in\Lambda^r_{\mathrm{loc,t,nt}}(\Lambda^1(M),\Lambda(M))$. Since T is nt, the map C defined by

$$C(X_0, ..., X_{r-1})=\begin{cases}T(\omega_0, ..., \omega_{r-1}) & \text{if } \forall i,\ X_i\in H_t(M), \rho(\omega_i)=X_i\,,\\ 0\,, & \text{if one of } X_i \text{ is in a chosen supplementary of } H_t(M)\,,\end{cases}$$

is well defined, and $T=\rho(C)$.

(ii) We denote by V_1 the space of all exact one-forms on M, by V_2 the space of all transversal one-forms on M and by V_3 a supplementary of V_1+V_2 in $\Lambda^1(M)$. Let C be in $\mathscr{A}^r_{\mathrm{loc,t,nc}}(N)$. We define T by

$$T(\omega_0, ..., \omega_{r-1})=\begin{cases}C(u_0, ..., u_{r-1}) & \text{if } \forall i,\ \omega_i=\mathrm{d}u_i\,,\\ 0 & \text{if one of } \omega_i\in V_2+V_3\,.\end{cases}$$

This construction is possible because if $\mathrm{d}u$ is transversal then $u\in I$ so $C(u, ...)=0$. It is clear that $T\in\Lambda_{\mathrm{loc,t,nt}}(\Lambda^1(M),N)$ and $\rho(T)=C$. □

Let Γ be a linear connection without torsion. We denote by ∇ the covariant derivative associated with Γ and by $L_X\nabla$ its Lie derivative in the direction of X. We consider the map

$$\Phi_\Gamma : H(M)\times H(M)\to \Lambda^2(M)\,,$$
$$(X_0, X_1)\to [(Y_0, Y_1)\to \tfrac{1}{2}\mathrm{tr}(L_{X_0}\nabla(Y_0)L_{X_1}\nabla(Y_1) - L_{X_0}\nabla(Y_1)L_{X_1}\nabla(Y_0))]\,,$$

and we define the local map S^3_Γ by

$$S^3_\Gamma(u, v) = \langle \Lambda, \rho(\Phi_\Gamma)(\mathrm{d}u, \mathrm{d}v)\rangle \quad \forall u, v\in N\,.$$

Finally we consider $T_\Gamma : H(M)^3 \to N$ defined by

$$T_\Gamma(X, Y, Z) = \frac{1}{3!}\underset{X,Y,Z}{\mathfrak{S}} \mathrm{tr}\, A^\nabla\{[A^\nabla(Y), A^\nabla(Z)] + 3R(X, Y)\}\,,$$

where A^∇ is the map $X\to [Y\to \nabla_X Y]$ and R is the curvature tensor of Γ. $\mathfrak{S}$ means sum over all cyclic permutations.

Proposition 6.4 [2,5,8].

(i) Φ_Γ is a non-exact cocycle for ∂'; moreover, its cohomology class is independent of Γ.

(ii) S^3_Γ is a non-exact cocycle for ∂; moreover its cohomology class is independent of Γ.

(iii) If Γ is adapted to the leaves then $\rho(\Phi_\Gamma)$, S^3_Γ, $\rho(T_\Gamma)$ and $T^2_\Gamma = \mu^*(T_\Gamma)$ are tangential. □

We denote by L_t the subalgebra of $H_t(M)$ of all vector fields X on M such that $L_X\Lambda = 0$ and by L^* the subalgebra of all Hamiltonian vector fields and we write $L_t = L^* \oplus L_S$.

Proposition 6.5. *Let Γ be a linear connection without torsion adapted to the leaves. Then*

(i) Every differentiable one-cocycle is one-differentiable.

(ii) Every tangential differentiable two-cocycle C has the form

$$C = aS^3_\Gamma + C_1 + \partial B\,, \quad a\in I,\ C_1 \in Z^2_{\text{1-diff,t,nc}}(N, \partial),\ B\in \mathscr{A}^0_{\text{diff,t,nc}}(N)\,.$$

(iii) Every tangential differentiable three-cocycle C has the form

$$C = S^3_\Gamma \wedge L_X + aT^2_\Gamma + C_1 + \partial B\,,$$
$$X\in L_t,\ a\in I,\ C_1 \in \mathscr{A}^2_{\text{1-diff,t,nc}}(N),\ B\in \mathscr{A}^1_{\text{diff,t,nc}}(N)\,.$$

Proof. The proof uses the classical method of symbolic calculus of ref. [4]. First we remark that the symbol of a tangential, differentiable cochain is a polynomial function of the variables ξ^j_i, $j \leqslant 2p$, where

$$\xi_i = \sum_{j=1}^{n} \xi^j_i \,\mathrm{d}x^j\,.$$

From the result of ref. [8], we deduce directly that, up to a correction by a tangential differentiable coboundary, each tangential differentiable cocycle has a symbol of the following form: the product of a polynomial function of $\Lambda(\xi_i, \xi_j)$ by a polynomial function of the variables ξ_i, with degree less than or equal to 1.

Now, the proof follows exactly the proof of ref. [5] in the symplectic case.

If the order of a two-cocycle is (3,3), its symbol coincides with the symbol of aS^3_Γ with $a \in I$. If that order is (2,2), its symbol is the symbol of a coboundary of a tangential, differentiable, nc cochain.

Moreover, if the order of a three-cocycle is (3,3,1), its symbol is the symbol of $S^3_\Gamma \Lambda L_X$, where X is in L_t, and if that order is (2,2,2), then either the symbol of our cocycle is the symbol of a coboundary or T^2_Γ is a cocycle and the symbol is the symbol of aT^2_Γ with $a \in I$.

Finally, in each other possible case, the symbol of a three-cocycle coincides with the symbol of a coboundary of a tangential, differentiable, nc cochain. □

Proposition 6.6. *For* $k \leqslant 3$,

$$H^k_{\mathrm{loc,t,nc}}(N, \partial) \cong H^k_{\mathrm{diff,t,nc}}(N, \partial) .$$

Proof. The proof of this proposition is very similar to the arguments in the symplectic case given in ref. [4]. □

Proposition 6.7.

$$Z^k_{\mathrm{loc,t,nc}}(N, \partial) = \rho^*(Z^k_{\mathrm{loc,t,nt}}(\Lambda^1(M), N)) \quad \text{for } k \leqslant 3 .$$ □

7. The maps Θ^τ_Λ

Let us denote by $\tau: \mathscr{A}_{\mathrm{loc,nc}}(N) \to \Lambda_{\mathrm{loc}}(\Lambda^1(M), N)$ an arbitrary right inverse of ρ^* over $\Lambda_{\mathrm{loc}}(\Lambda^1(M), N)$. Let $U, x^1, \ldots, x^n$ be a natural chart of M, and let us consider the two-form defined on U by

$$F_U = \sum_{i=1}^{p} \mathrm{d}x^i \wedge \mathrm{d}x^{i+p} .$$

Since F_U is closed there exists a one-form ω_U on U such that $\mathrm{d}\omega_U = F_U$. We define the map

$$\Theta^\tau_\Lambda : \mathscr{A}_{\mathrm{loc,nc}}(N) \times \mathscr{A}_{\mathrm{loc,nc}}(N) \to \mathscr{A}_{\mathrm{loc,nc}}(N) ,$$

$$\Theta^\tau_\Lambda(A, B)_{/U} = \rho^* i(\omega_U)\tau[A, B] - (-1)^b[\rho^* i(\omega_U)\tau A, B] - [A, \rho^* i(\omega_U)\tau B] ,$$

$$\forall A \in \mathscr{A}_{\mathrm{loc,nc}}(N),\ \forall B \in \mathscr{A}^b_{\mathrm{loc,nc}}(N) .$$

The definition makes sense because $\Theta^\tau_\Lambda(A, B)$ does not depend on the choice of ω_U in U.

Proposition 7.1. $\forall A\in\mathscr{A}^a_{\mathrm{loc,nc}}(N)$, $\forall B\in\mathscr{A}^b_{\mathrm{loc,nc}}(N)$, $\forall C\in\mathscr{A}^c_{\mathrm{loc,nc}}(N)$

(i) $$\Theta^\tau_\Lambda(A, B)\in\mathscr{A}^{a+b-1}_{\mathrm{loc,nc}}(N)\ ,$$

(ii) $$\Theta^\tau_\Lambda(A, B)=(-1)^{ab+1}\Theta^\tau_\Lambda(B, A)\ ,$$

(iii) $$\underset{a,b,c}{\mathfrak{S}}\,(-1)^{ac}(\Theta^\tau_\Lambda([A, B], C)-[A, \Theta^\tau_\Lambda(B, C)])=0\ .$$ □

We construct from Θ^τ_Λ the operator

$$D^\tau: \mathscr{A}_{\mathrm{loc,nc}}(N)\to\mathscr{A}_{\mathrm{loc,nc}}(N), A\to\Theta^\tau_\Lambda(A, P)\ .$$

Proposition 7.2. *Let $(U, x^1, ..., x^n)$ be a natural chart of M and let ω be a one-form on U such that $\mathrm{d}\omega=F_U$. Then*

(i) $$\forall u\in N,\ [\rho(\omega), H_u]=H_{\rho(\omega)u}-H_u\ ,\quad H_u=\rho(\mathrm{d}u)\ .$$

(ii) *If $T\in\mathbb{B}\cap\Lambda^c(\Lambda^1(M), N)$ and $T=\rho^*(C)$, then*

$$L_{\rho(\omega)}\rho^*(T)=\rho^*(\rho(L_{\rho(\omega)}C))-c\rho^*(T)\ .$$

(iii) $$L_{\rho(\omega)}F_U(H_u, H_v)=-F_U(H_u, H_v)=-\Lambda(\mathrm{d}u, \mathrm{d}v)\ \forall u, v\in N\ .$$

(iv) $$L_{\rho(\omega)}\circ\partial=\partial\circ L_{\rho(\omega)}-\partial\ .$$

Proof.

(i) For every X tangent to the leaves,

$$F_U(H_{\rho(\omega)u}, X)=-\mathrm{d}(\rho(\omega)u)(X)=-L_{\rho(\omega)}\mathrm{d}u\cdot(X)=L_{\rho(\omega)}i(H_u)F_U\cdot(X)$$
$$=F_U([\rho(\omega), H_u], X)+L_{\rho(\omega)}F_U(H_u, X)\ ,$$
$$L_{\rho(\omega)}F_U=\mathrm{d}i(\rho(\omega))F_U=F_U\ .$$

Since every vector field occurring in this relation is tangent to the leaves, we deduce the formula.

(ii) Follows immediately from (i).

(iii) Observe that $L_{\rho(\omega)}\Lambda=-P$. □

Proposition 7.3. *Let τ be a right inverse of ρ^* such that*

$$\tau(\mathscr{A}_{\mathrm{loc,t,nc}}(N))=\Lambda_{\mathrm{loc,t,nt}}(\Lambda^1(M), N)\ ,$$

and let $(U, x^1, ..., x^n)$ be a natural chart of M and ω a one-form on U such that

$d\omega = F_U$. *Then*

$$D^\tau A = -(a+1)A + \rho^* i(\omega)(\partial''\tau - \tau\partial)A \quad \forall A \in \mathscr{A}^a_{\mathrm{loc,t,nc}}(N) .$$

Proof. $\forall A \in \mathscr{A}^a_{\mathrm{loc,t,nc}}(N)$,

$$D^\tau A = \rho^* i(\omega)\tau[A, P] + [\rho^* i(\omega)\tau A, P] - [A, \rho^* i(\omega)\tau P] ,$$

since $\tau P = \Lambda$ and $\rho^* i(\omega)\tau P = L_{\rho(\omega)}$. Then

$$\begin{aligned} D^\tau A &= (L_{\rho(\omega)} - \partial\rho^* i(\omega)\tau - \rho^* i(\omega)\tau\partial)A \\ &= \rho^* i(\omega)(\partial''\tau - \tau\partial)A + (L_{\rho(\omega)} - \rho^*\partial'' i(\omega)\tau - \rho^* i(\omega)\partial''\tau)\rho^*(\tau A) . \end{aligned}$$

But if $\tau A = \rho(C)$ then

$$\begin{aligned} &(L_{\rho(\omega)} - \rho^*\partial'' i(\omega)\tau - \rho^* i(\omega)\partial''\tau)(\rho^*(\tau A)) \\ &= \rho^* \circ \rho(L_{\rho(\omega)} C - \partial' i(\rho(\omega))C - i(\rho(\omega))\partial'\tau) - (a+1)A \\ &= -(a+1)A . \end{aligned}$$

□

Proposition 7.4. *There exists a right inverse* $\tau: \mathscr{A}_{\mathrm{loc,t,nc}}(N) \to A_{\mathrm{loc,t,nt}}(\Lambda^1(M), N)$ *of* ρ^* *such that*

(i) $\tau \circ \rho^* = 1$ *on* $T(M)$, *the space of all antisymmetric contravariant tensors on* M;

(ii) $\rho^* i(\omega)(\partial''\tau - \tau\partial) = 0$ *on* $Z^p_{\mathrm{loc,t,nc}}(N, \partial)$ *for* $p \leqslant 3$ *and* $B_{\mathrm{loc,t,nc}}(N, \partial)$;

(iii) $\rho^* i(\omega)(\partial''\tau - \tau\partial) = -1$ *on* IS^3_Γ *and* $S^3_\Gamma \Lambda L_S$.

Proof. For $p = 1$ or $p > 3$ we decompose $\mathscr{A}^{p-1}_{\mathrm{loc,t,nc}}(N)$ as follows:

$$\mathscr{A}^{p-1}_{\mathrm{loc,t,nc}}(N) = \rho^*(T_t(M)) \oplus \rho^*(\partial'' E) \oplus \rho^*(F) ,$$

where

$$\rho^*(B^p_t(M)) \oplus \rho^*(\partial'' E) = \rho^*(B^p_{\mathrm{loc,t,nt}}(\Lambda^1(M), N)) ,$$

and where $T_t(M)$ is the space of all tangential tensors on M and $B^p_t(M)$ is $\partial''(T^{p-1}_t(M))$. We consider

ψ, a right inverse of $\rho^*: E \to \rho^*(E)$,

σ, a right inverse of $\partial: \rho^*(E) \to \partial(\rho^*(E))$,

τ_2, a right inverse of $\rho^*: F \to \rho^*(F)$.

For $p = 2$ or 3 we replace $\rho^*(F)$ by $IS^3_\Gamma \oplus \rho^*(F')$ or $(S^3_\Gamma \Lambda L_S) \oplus \rho^*(F')$, and we choose τ_2 such that

$$\tau_2(S^3_\Gamma) = \langle \Lambda, \rho(\Phi_\Gamma) \rangle ,$$

$$\tau_2(S^3_\Gamma \Lambda L_X) = \langle \Lambda, \rho(\Phi_\Gamma) \rangle \Lambda L_X ,$$

$$\tau_2(a S^3_\Gamma) = a\tau_2(S^3_\Gamma) \quad \forall a \in I .$$

We define τ by

(i) the right inverse of $\rho^*: T_\mathrm{t}(M) \to \rho^*(T_\mathrm{t}(M))$ on $\rho^*(T_\mathrm{t}(M))$,

(ii) $\partial'' \circ \psi \circ \sigma$ on $\rho^*(\partial'' E)$,

(iii) τ_2 on $\rho^*(F)$. □

Remark. The above constructed τ is onto, thus it satisfies the hypothesis of proposition 7.3.

Proposition 7.5. *For D^τ, the following identities hold:*

(1) $D^\tau \circ \partial = \partial \circ D^\tau - \partial$,

(2) $D^\tau + k = 0$ *on* $B^k_{\mathrm{loc,t,nc}}(N, \partial)$,

(3) $D^\tau + 1 = 0$ *on* $Z^1_{\mathrm{loc,t,nc}}(N, \partial)$,

(4) $(D^\tau + 2)(D^\tau + 3) = 0$ *on* $Z^2_{\mathrm{loc,t,nc}}(N, \partial)$,

(5) $(D^\tau + 3)(D^\tau + 4) = 0$ *on* $Z^3_{\mathrm{loc,t,nc}}(N, \partial)$,

(6) $(D^\tau + 1)^2 = 0$ *on* $\mathscr{A}^0_{\mathrm{loc,t,nc}}(N)$,

(7) $(D^\tau + 2)^2 (D^\tau + 3) = 0$ *on* $\mathscr{A}^1_{\mathrm{loc,t,nc}}(N)$,
$(D^\tau + 2)^2 = 0$ *on* $\mathscr{A}^1_{\mathrm{1\text{-}diff,t,nc}}(N)$,

(8) $(D^\tau + 3)^2 (D^\tau + 4) = 0$ *on* $\mathscr{A}^2_{\mathrm{loc,t,nc}}(N)$,
$(D^\tau + 3)^2 = 0$ *on* $\mathscr{A}^2_{\mathrm{1\text{-}diff,t,nc}}(N)$. □

8. Existence of tangential formal deformations of (*N*, *P*)

Let us define $D_\nu : N_\nu \to N_\nu$,

$$\sum_{k \geqslant 0} \nu^k u_k \to \sum_{k \geqslant 1} k \nu^{k-1} u_k .$$

The same arguments as in ref. [3] prove:

Proposition 8.1. *The equation*

$$(\nu D_\nu + 1) L_\nu + \tfrac{1}{2} \Theta^\tau_\Lambda (L_\nu, L_\nu) = 0$$

admits a unique solution L_ν such that

$$L_0 = P ,$$

$$L_1 = \rho^*(T) + \partial E , \quad T \in Z^2_{\mathrm{loc,t}}(\Lambda(M), N) \cap T_\mathrm{t}(M) ,$$

$$E \in \mathscr{A}^0_{\mathrm{loc,t,nc}}(N) ,$$

$$L_2 = -\tfrac{1}{2}(1 + D^\tau) \Theta^\tau_\Lambda (L_1, L_1) + a S^3_\Gamma , \quad a \in I .$$

(a) This solution is a tangential formal deformation of (N, P).

(b) If $a=0$ and $E=0$ this deformation is one-differentiable.
(c) If $L_1=0$ then $\forall k$, $L_{2k+1}=0$, and $L'_\nu=\sum_{k\geqslant 0}\nu^k L_{2k}$ is the unique solution of

$$(2\nu D_\nu+1)L'_\nu+\tfrac{1}{2}\Theta^\tau_\Lambda(L'_\nu, L'_\nu)=0\,,$$
$$L'_0=P, \qquad L'_1=aS^3_\Gamma\,. \qquad \square$$

As in the symplectic case we introduce multiparametric deformations and we deduce the following proposition.

Proposition 8.2 [3]. *Every tangential formal deformation of order $k\geqslant 0$ of (N, P) extends to a tangential formal deformation of (N, P).* □

9. Existence of tangential star products on a regular Poisson manifold

From the study of the star product on a symplectic manifold we easily deduce that the term M_3 of a tangential star product has the form

$$\frac{1}{3!}S^3_\Gamma+\rho^*(T)+\partial E\,, \quad T\in Z^2_{\mathrm{loc,t}}(\Lambda(\Lambda^1(M),N))\cap T_\mathrm{t}(M)\,,$$
$$E\in\mathscr{A}^0_{\mathrm{loc,t,nc}}(N)\,.$$

Proposition 9.1. *A tangential formal deformation $L_\nu=\sum_{k\geqslant 0}\nu^k L_k$ of (N, P) derives from a tangential weak star product if and only if*

$$L_1=\frac{1}{3!}S^3_\Gamma+\rho^*(T)+\partial E\,, \quad T\in Z^2_{\mathrm{loc,t}}(\Lambda(\Lambda^1(M),N))\cap T_\mathrm{t}(M)\,,$$
$$E\in\mathscr{A}^0_{\mathrm{loc,t,nc}}(N)\,.$$

Proof. The form of the term M_3 of a tangential star product and that of a tangential two-cocycle being known, the proof of this proposition is identical to the symplectic case treated in ref. [3]. □

Proposition 9.2 [3]. *Every tangential star product or tangential weak star product of order $2k$ is the driver of a tangential star product or a tangential weak star product.* □

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